LCM and GCD Relationship Proof

12/25/2023

Theorem. *The following relationship between the Least Common Multiple (LCM) and the Greatest Common Divisor (GCD) of* $a, b \in \mathbb{N}$ *holds:*

$$LCM(a, b) = \frac{ab}{GCD(a, b)} \tag{1}$$

Proof. Let $p_1, p_2, \ldots, p_n$ be all of the prime numbers appearing in at least one of the prime factorization of a or the prime factorization of b, and let

$$a = p_1^{i_1} p_2^{i_2} ... p_n^{i_n}$$

$$b = p_1^{j_1} p_2^{j_2} ... p_n^{j_n}$$

where some of the i_k, j_k terms may be zero. We prove this with two lemmas.

Lemma 1. *Some* $c \in \mathbb{N}$ *is a common multiple of* a, b *if and only if it takes the form:*

$$c = p_1^{h_1} p_2^{h_2} ... p_n^{h_n} q$$

with each $h_k \geq MAX\{i_k, j_k\}$*, and where* q *is the (possibly empty) product of all prime factors of* c *not equal to* $p_1, p_2, \ldots, p_n$*.*

Proof. Exercise. □

The LCM, then, is obtained by setting $q = 1$ and setting each h_k equal to $MAX\{i_k, j_k\}$. So we have:

$$LCM(a, b) = p_1^{MAX\{i_1, j_1\}} p_2^{MAX\{i_2, j_2\}} ... p_n^{MAX\{i_n, j_n\}}$$

Now onto the second lemma.

Lemma 2. *Some* $d \in \mathbb{N}$ *is a common divisor of* a, b *if and only if it takes the form:*

$$c = p_1^{g_1} p_2^{g_2} ... p_n^{g_n}$$

with each $g_k \leq MIN\{i_k, j_k\}$*.*

Proof. Exercise. □

The GCD, then, is obtained by setting each g_k equal to $MIN\{i_k, j_k\}$. This gives:

$$GCD(a,b) = p_1^{MIN\{i_1,j_1\}} p_2^{MIN\{i_2,j_2\}} ... p_n^{MIN\{i_n,j_n\}}$$

Now we take this home. Since $p_k^{MAX\{i_k,j_k\}} p_k^{MIN\{i_k,j_k\}} = p_k^{i_k} p_k^{j_k}$ regardless of which of i_k, j_k is the maximum or the minimum (or both), we have:

$$\begin{aligned}
LCM(a,b)GCD(a,b) &= (p_1^{MAX\{i_1,j_1\}} p_2^{MAX\{i_2,j_2\}} ... p_n^{MAX\{i_n,j_n\}})(p_1^{MIN\{i_1,j_1\}} p_2^{MIN\{i_2,j_2\}} ... p_n^{MIN\{i_n,j_n\}}) \\
&= (p_1^{MAX\{i_1,j_1\}} p_1^{MIN\{i_1,j_1\}})(p_2^{MAX\{i_2,j_2\}} p_2^{MIN\{i_2,j_2\}}) ... (p_n^{MAX\{i_n,j_n\}} p_n^{MIN\{i_n,j_n\}}) \\
&= (p_1^{i_1} p_1^{j_1})(p_2^{i_2} p_2^{j_2}) ... (p_n^{i_n} p_n^{j_n}) \\
&= (p_1^{i_1} p_2^{i_2} ... p_n^{i_n})(p_1^{j_1} p_2^{j_2} ... p_n^{j_n}) \\
&= ab
\end{aligned}$$

This then gives:

$$LCM(a,b) = \frac{ab}{GCD(a,b)}$$

as desired.

□